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Integral Approach to an Approximate Analysis of Thrust Vector Control by Secondary Injection

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An approximate analysis of thrust vector control by secondary fluid injection is approached through the application of the integral form of the conservation laws and the equation of state for a mixture of gases. The thrust augmentation and the side force are expressed in terms of the flow conditions at the exit section of the nozzle and the problem is thus reduced to that of determining these conditions. In this sense the present approach is different from the usual one where the pressure distribution over the nozzle surface is the object of the analysis. Considering inert gases, the necessary equations are developed and the steps involved in obtaining a solution are discussed. An approximate formula for the side force, applicable under certain conditions of operation, is obtained. Results given by the formula are compared and found to be in agreement with appropriate experimental results.

Nomenclature

A = area
c = specific heat at constant pressure
d = diameter of nozzle exit
E = energy per unit length
$\mathbf{F}$ = force
δF_a = thrust augmentation
F_s = side force
h = enthalpy per unit mass
H = total or stagnation enthalpy per unit mass
$\mathbf{i}$ = unit vector in the direction of X axis
$\mathbf{j}$ = unit vector in the direction of Y axis

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J = a constant
l = distance from injection port to nozzle exit along the nozzle surface in the plane of injection
$\dot{m}$ = mass flow rate
$\mathbf{n}$ = outward normal to nozzle surface
p = pressure
R = radius; also gas constant
$\mathcal{R}$ = region of space enclosed by the throat and exit sections and the nozzle surface between them
S = area of nozzle surface influenced by secondary injection
u = X component of exit velocity in the presence of injection
u' = difference of u and U_e
U_e = X component of exit velocity in the absence of injection
v = Y component of exit velocity
$\mathbf{V}$ = velocity
γ = ratio of specific heats
ϵ = expansion ratio
ζ = distance along the nozzle surface in the plane of injection from injection point
μ = molecular weight
ρ = density
ϕ = a dimensionless number

Subscripts

e = nozzle exit
i = injectant
0 = condition in the absence of injection
S = shock
∞ = primary flow condition at injection station in the absence of injection

Introduction

THE problem of thrust vector control by secondary fluid injection is not readily amenable to theoretical analysis. This situation arises mostly out of the inability to take account of some of the important details that concern the complex interaction occurring between the secondary fluid and the primary flow in the rocket nozzle. Attempts, however, have been made at analyzing the problem on the basis of some simple assumptions. A detailed review of these attempts is given in Ref. 1. In most of these investigations the following flow field is assumed. The secondary fluid is assumed to form a body of some shape, and associated with the body is a shock wave (strong or weak) in the primary flow. A choice is then made whether to consider the interaction of the shock wave and the boundary layer along the nozzle wall and the possible separation of the boundary layer. Based on this choice, a disturbed region on the nozzle wall is defined. If shock/boundary-layer interaction is not considered, the disturbed wall region is simply that bounded by the shock; but if the shock/boundary-layer interaction is considered, the disturbed region is usually assumed to be made up of the region in which separation occurs and the region immediately behind the shock, in which pressure increases. Criteria have yet to be established to indicate when to make one or the other choices for the disturbed wall region.

Once the flow model has been decided, the analysis proceeds to determine the disturbed region and the pressure distribution over that region. Neither the body shape, nor the shock shape, nor the separated region when existing are known a priori. The analysis of the problem as posed is, therefore, carried out in an approximate manner. Finally, the induced force is derived from an integration of the excess (over the original) pressure forces acting over the disturbed region.

It is apparent that, in such analysis, assumptions have to be made with regard to important but meagerly understood aspects of the flow field near the injection port, and the final result hinges heavily on how closely the theoretically derived disturbance region and the pressure distribution in this region approximate the actual situation. Without having more information than is presently available with regard to the actual flow field in the vicinity of injection port, one cannot expect that these approximate determinations of the disturbed region and the pressure field over it are sufficiently accurate. The theoretical pressure distribution over the nozzle wall, particularly in the vicinity of the injection port, is sensitive to the assumed flow model. Therefore, to obtain the induced force by integration of the theoretically derived pressure forces on the nozzle wall would appear unsatisfactory.

In the light of the preceding observations, it is fruitful to approach the problem in a manner that does not require a knowledge of many important details of the flow structure and does not involve the integration of pressures over the nozzle wall and yet is sufficiently sound. Such an approach is given in this paper. The basic idea of the approach is to reduce the calculation of the induced force to that of the flow conditions at the exit section of the nozzle. The purpose of this paper is to formulate the problem according to this approach and to discuss the features involved in obtaining a solution.

Equations

Consider the case of injection through a single port (Fig. 1). The difference between the force $\mathbf{F}$ on the rocket in the presence of secondary fluid injection and the force $\mathbf{F}_0$ on the rocket in the absence of injection is given by

$$\mathbf{F} - \mathbf{F}_0 = \iint_S (p - p_0)\mathbf{n}\, dS + \iint_{A_i} [\mathbf{V}(\rho\mathbf{V}\cdot\mathbf{n}) + (p - p_0)\mathbf{n}]dS \quad (1)$$

where $\mathbf{n}$, as shown in Fig. 1, is the outward normal to the surface of the rocket nozzle; A_i is the surface area of the injection hole; and S is the disturbed area of the nozzle surface, i.e., the extent of this surface affected by the processes resulting from the secondary injection. In Eq. (1) the flow quantities with the subscript 0 refer to conditions in the absence of injection, and quantities without a subscript refer to conditions in the presence of injection. (This scheme will be adopted generally in the rest of the paper.) The integral over the surface S is the induced force, and the integral over the area A_i is the reaction force. Introduce a Cartesian coordinate system X, Y, Z, such that the X axis is disposed of, as shown in Fig. 1, along the axis of the nozzle, and the X-Y plane contains the center of the injection hole. Then, the thrust augmentation δF_a and the side force F_s on the rocket are given by

$$\delta F_a = -\mathbf{i}\cdot(\mathbf{F} - \mathbf{F}_0)$$
$$F_s = -\mathbf{j}\cdot(\mathbf{F} - \mathbf{F}_0)$$

As a first step in the approach suggested in this paper, the force components δF_a and F_s are expressed in terms of the flow conditions at the exit section of the nozzle. To do this, mark out a region of space $\mathcal{R}$, enclosed by a fixed surface formed by the throat and exit planes (normal to X axis) of the nozzle and the surface of the nozzle wall between those planes (see Fig. 1). Considering steady flow an equation for

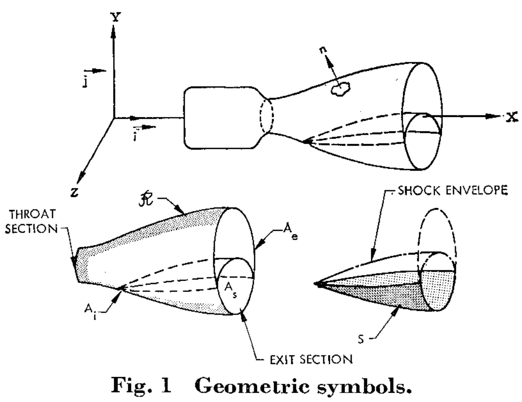

Fig. 1 Geometric symbols.

the conservation of momentum of the fluid enclosed in the region $\mathcal{R}$ when no secondary injection is present is obtained. Similarly, another equation for the conservation of momentum of the fluid enclosed in the region $\mathcal{R}$ when secondary injection is present is obtained. Subtracting one from the other of the equations obtained, it is found that

$$\delta F_a = \iint_{A_S} pdS - p_e A_S + \iint_{A_S} (\rho \mathbf{V}\cdot\mathbf{i})\mathbf{V}\cdot\mathbf{i}\, dS - \dot{m}_0 U_e \frac{A_S}{A_e} \tag{2}$$

$$F_s = \iint_{A_S} \mathbf{j}\cdot\mathbf{V}(\rho\mathbf{V}\cdot\mathbf{i})\, dS \tag{3}$$

where A_S is that portion of A_e, the exit cross section of the nozzle, which is influenced by the secondary injection; $\dot{m}_0$ is the rate of mass flow of the primary fluid; and the subscript e denotes the conditions at the exit section in the absence of injection. It has been assumed as usual that $\mathbf{V}_e$ is uniform over A_e and equal to $\mathbf{i}\ U_e$. In obtaining Eqs. (2) and (3), the viscous forces acting on the boundary of region $\mathcal{R}$ are neglected. With this understanding, these equations take into account the boundary layer along the nozzle wall and its interaction with the induced shock wave. It is, however, assumed that the secondary injection and the consequent shock/boundary-layer interaction do not affect the flow conditions at the throat section of the nozzle.

In Eqs. (2) and (3), the thrust augmentation and the side force are expressed in terms of the flow conditions at the exit plane of the nozzle. The right-hand members of these equations, however, contain A_S and p, ρ, and $\mathbf{V}$ over A_S as unknowns. Thus, to be able to solve for δF_a and F_s, one needs further equations. To obtain the additional equations, the principles of conservation of mass and energy and the equation of state are used. For simplicity and for the ease of discussing the utility of the present approach in light of available experimental results, only the case where the primary and secondary fluids are nonreacting gases is considered, and heat conduction and viscous dissipation are neglected. Applying the principles of conservation of mass and energy to the fluid in the region $\mathcal{R}$ for the two cases of with and without injection one obtains the following two relations:

$$\iint_{A_S} (\rho\mathbf{V}\cdot\mathbf{i})\, dS = \dot{m}_i + \dot{m}_0 \frac{A_S}{A_e} \tag{4}$$

$$\iint_{A_S} \rho\left(h + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{i}\, dS = \dot{m}_i H_i + \dot{m}_0 H_0 \frac{A_S}{A_e} \tag{5}$$

Here, $\dot{m}_i$ is the rate of mass flow (into the rocket nozzle) of the secondary fluid; H_0 is the total (or stagnation) enthalpy, per unit mass, of the primary fluid; and H_i is the total enthalpy, per unit mass, of the secondary fluid. Next, it is assumed that the primary and secondary fluids are perfect gases and that their mixture follows an equation of state, of the form

$$h = (c/R)(p/\rho) \tag{6}$$

where c and R are, respectively, the appropriate specific heat at constant pressure and the gas constant per unit mass. In obtaining Eqs. (4) and (5), it is assumed that the conditions of flow in the presence of injection are uniform over the area A_i.

Equations (2) and (6) form a system of five equations for eight unknowns and yet are not in a form suitable for constructing a solution of our problem. The difficulty is that, in the presence of secondary injection, the flow quantities on the surface A_S are, in general, nonuniform. To be able to proceed further, it is assumed that, even in the presence of secondary injection, uniform conditions exist‡ on the surface area A_S, these conditions being, in general, different from those that occur on the surface area $A_e - A_S$, i.e., those that occur on A_e in the absence of secondary injection. It is observed that there is no a priori reason to expect that, in the presence of injection, conditions on A_S would be uniform. It is, however, conceivable that if the injection port is sufficiently ahead of the exit plane of the nozzle and if there is sufficient mixing of the primary and secondary fluids, the flow conditions on A_S may be nearly uniform. Using this assumption, and denoting by u and v the x and y components of $\mathbf{V}$ over A_S, Eqs. (2–6) may be rewritten as

$$\delta F_a = (p - p_e)A_S + \dot{m}u - \dot{m}_0\, U_e(A_S/A_e) \tag{7}$$

$$F_s = \dot{m}v \tag{8}$$

$$\rho u\, A_S \equiv \dot{m} = \dot{m}_i + \dot{m}_0(A_S/A_e) \tag{9}$$

$$\dot{m}\left(h + \frac{u^2 + v^2}{2}\right) = \dot{m}_i H_i + \dot{m}_0\, H_0 \frac{A_S}{A_e} \tag{10}$$

$$h = (c/R)\ (p/\rho) = (c/R)(pA_S u/\dot{m}) \tag{11}$$

These equations form the basis for an approximate solution of the problem. They constitute, however, a system of only five equations for eight unknowns, viz., δF_a, F_s, A_S and the fluid quantities p, ρ, h, u, v assumed uniform over A_S. It is suggested that the required additional relations between the fluid quantities at the exit section may be developed, to sufficient accuracy, on the basis of some over-all experimental and theoretical studies without seeking the complex details of the interaction between the secondary and primary fluids. In the next section an example is given of how this may be done and how an approximate solution of the system of Eqs. (7–11) suitable for certain conditions of operation may be constructed. For a comprehensive development of the additional relations, one needs more experimental information than is presently available.

An Approximate Solution

Consider first the area A_S. To calculate this area one needs to know the shape of the shock wave induced by the secondary injection. A sufficiently accurate determination of the shape of the complete shock wave (particularly of the portion near the injection port) may require many of the details of the flow field near the injection port and may not be readily feasible. In this approach to the problem the shape of the complete shock is not of interest, much less is its shape near the injection port. What one needs to know is the shape of the shock at the exit section of the nozzle. If the injection port is located at a sufficient§ distance from the exit section it is expected that the shape of the shock at the exit section may be computed sufficiently accurately by a method that will not involve many details of the actual mechanics of the interaction between the secondary and primary fluids. An example of the method in question is the attempt by Broadwell[2] to apply the results of blast wave theory to the analysis of thrust vector control by secondary injection. Following Broadwell, the results of blast wave theory will be used in the next section to compute the shock shape at the exit section. For the present, to proceed with the solution of this problem, A_S is treated as an independent quantity. Next considered is the possibility of expressing p or u in terms of the other unknowns. After examining several possibilities in light of

‡ One can improve this assumption by introducing suitably defined mean values; however, for simplicity, uniformity is assumed in the following.

§ Criteria for what is meant by sufficient distance to be established on the basis of appropriate experimental and theoretical investigations.

the limited experimental information presently available, the following scheme is adopted. Write

$$u = U_e + u'$$

and express Eqs. (7) and (11) as

$$\delta F_a = (p - p_e)A_S + \dot{m}_0 U_e \left(\frac{\dot{m}_i}{\dot{m}_0} + \frac{\dot{m}_i}{\dot{m}_0}\frac{u'}{U_e} + \frac{A_S}{A_e}\frac{u'}{U_e}\right) \quad (7a)$$

$$h = \frac{c}{R}\frac{p A_S}{\dot{m}} U_e\left(1 + \frac{u'}{U_e}\right) \quad (11a)$$

The term u^2 in the energy equation (10) becomes

$$u^2 = U_e^2\left[1 + 2\frac{u'}{U_e} + \left(\frac{u'}{U_e}\right)^2\right]$$

Now it is assumed that $u' \ll U_e$ and that $u' < v$. Then Eqs. (7a) and (11a) may be rewritten in the approximate form

$$\delta F_a \simeq (p - p_e)A_S + \dot{m}_i U_e \quad (7b)$$

$$h \simeq (c/R)(pA_S U_e/\dot{m}) \quad (11b)$$

and the term u^2 in the energy equation in the form

$$u^2 \simeq U_e^2 + 2u' U_e$$

To eliminate u' from the problem and proceed to a solution, set arbitrarily (for lack of suitable experimental information) that $2u'U_e \simeq 2vU_e$, and write

$$u^2 \simeq U_e^2 + 2vU_e \quad (12)$$

It is seen that once p is known, the thrust augmentation is immediately given by Eq. (7b).

Another relation is still needed. One may try to stipulate a relation for p, but due to insufficient experimental information it is difficult to do so at present. Therefore, leave p as is and solve for the side force in terms of it. This procedure is satisfactory for, as will be seen later, it will enable one to check the calculated formula for the side force (or equivalently the utility of the present approach) with experimental results. With this understanding and using the relations (11b) and (12), Eqs. (8–11) may be solved to obtain the side force as

$$F_S = -\left(\dot{m}_i + \dot{m}_0\frac{A_S}{A_e}\right) + \left\{2\left(\dot{m}_i + \dot{m}_0\frac{A_S}{A_e}\right)\left(\dot{m}_i H_i + \dot{m}_0 H_0 \frac{A_S}{A_e} - \frac{c}{R} pA_S U_e\right)\right\}^{1/2} \quad (13)$$

In nondimensional form, this may be rewritten as

$$\frac{F_s}{\dot{m}_0 U_e} = -\left(\frac{\dot{m}_i}{\dot{m}_0} + \frac{A_S}{A_e}\right) + \left\{2\left(\frac{\dot{m}_i}{\dot{m}_0} + \frac{A_S}{A_e}\right) \times \left(\frac{\dot{m}_i}{\dot{m}_0}\frac{H_i}{U_e^2} + \frac{H_0}{U_e^2}\frac{A_S}{A_e} - \frac{c}{R}\frac{p A_S}{\dot{m}_0 U_e}\right)\right\}^{1/2} \quad (14)$$

where A_S and p are yet to be expressed explicitly in terms of the given parameters of the problem. As mentioned before, an approximate relation for A_S will be constructed while leaving p as is.

Area A_S under Certain Conditions

The area A_S is computed by applying the results of blast wave theory. In the absence of suitable experiments, it is difficult at present to assess the limits of applicability of such a computation. On the basis of Broadwell's preliminary work,[2] however, it may be expected that such a computation of A_S may be satisfactory under certain flow conditions.

Following Broadwell,[2] the shape of the induced shock wave is calculated now by using the results of the analogy between the unsteady flow field of a cylindrical blast wave and the steady, axisymmetric, inviscid, small disturbance, hypersonic flow field past a blunt-nosed slender body of revolution.‖ According to this analogy, the shape of the axisymmetric shock for the flow over the body is given by the relation

$$R = \left(\frac{2\gamma}{\pi J}\frac{E}{\rho_\infty}\right)^{1/4}\left(\frac{\zeta}{U_\infty}\right)^{1/2} \quad (15)$$

where, as shown in Fig. 2a, $R(\zeta)$ is the radius of the shock wave at the station ζ measured from the tip of the nose; U_∞ and ρ_∞ are the velocity and density, respectively, in the freestream ahead of shock E, which has the dimensions of energy per unit length and is set equal to the nose drag; γ is the ratio of the specific heats; and J is a constant whose value depends on the value of γ.#

In applying the result (15) to the problem of the shock induced by secondary injection into the flow through a rocket nozzle, Broadwell assumed that the shock in the nozzle is axisymmetric and of the same shape as the shock generated by secondary injection through a plane in an originally uniform supersonic stream (see Fig. 2b), the plane being parallel to the stream. The velocity and density of the undisturbed stream in this model are assumed to be the same as the velocity U_∞ and density ρ_∞ that exist at the injection station in the nozzle flow when there is no secondary injection. It is further assumed that the shock shape in this model (i.e., of Fig. 2b) may be computed by the relation (15). To connect E, the energy per unit length, to the injection parameters, Broadwell argued that $E/2$ is equal to the magnitude of the axial momentum the secondary fluid gains eventually, i.e., after it mixes fully with the primary fluid. For normal injection, this gain in momentum is assumed to be $\dot{m}_i U_\infty$ (recall that $\dot{m}_i$ is the mass flow rate of the injectant). Thus, setting E equal to $2\dot{m}_i U_\infty$ in Eq. (15), one obtains

$$R = \left(\frac{4\gamma}{\pi J}\frac{\dot{m}_i}{\rho_\infty U_\infty}\right)^{1/4}\zeta^{1/2} \quad (16)$$

where R is now the radius of the shock induced by secondary injection in the plane, and ζ is the length measured from the injection port along the axis of the shock (see Fig. 2b).

Denoting by A_∞ the cross-sectional area of the nozzle at the injection location, one has $\rho_\infty U_\infty A_\infty = \dot{m}_0$. Therefore, Eq. (16) may be rewritten as

$$R = \left(\frac{4\gamma}{\pi J}\frac{\dot{m}_i}{\dot{m}_0}A_\infty\right)^{1/4}\zeta^{1/2} \quad (17)$$

Equation (16) is used to compute the area A_S. To do this, it is assumed that the induced shock in the rocket nozzle is axially symmetric about the curve of intersection between the injection plane and the nozzle surface. The coordinate ζ is then measured along this line. With this definition of ζ and using Eq. (17), one obtains the following approximate expression for A_S:

$$\frac{A_S}{A_e} = \frac{4}{\pi}\left(\frac{R_S}{d}\right)^2 \sin^{-1}\phi + \frac{1}{\pi}\sin^{-1} 2\left(\frac{R_S}{d}\right)\phi - \frac{4}{\pi}\left(\frac{R_S}{d}\right)^3\phi - \frac{4}{\pi}\left(\frac{R_S}{d}\right)\left[\frac{1}{2} - \left(\frac{R_S}{d}\right)^2\right]\phi \quad (18)$$

‖ In this paper, the use of blast wave analogy is different from that of Broadwell's. Here, it is being used only to compute the shock shape at the exit section of the nozzle. Broadwell, on the other hand, uses it to analyze thrust vector control in its entirety, i.e., to compute the shape of the complete shock wave so as to obtain the disturbed area of the nozzle surface and to compute pressure distribution over that area. The blast wave analogy does not yield satisfactory results for the complete shock shape and the pressure distribution near the injection port.

Value of J varies from 1.55 for γ equal to 1.2; to 0.59 for γ equal to 1.667.

where d is the diameter of the exit section of the nozzle,

$$\phi = [1 - (R_S/d)^2]^{1/2}$$

and

$$\frac{R_S}{d} = \left(\frac{\gamma}{J}\right)^{1/4}\left(\frac{\dot{m}_i}{\dot{m}_0}\frac{A_\infty}{A_e}\right)^{1/4}\left(\frac{l}{d}\right)^{1/2} \tag{19}$$

with R_S being the radius of the shock at l, which is the distance from the injection port to the exit section along ζ.

Equations (17) and (18) are strictly applicable to the situation where the shock shape at the exit section of the nozzle is not dependent on the interaction between the shock and the boundary layer along the nozzle wall. This may be the case when the interaction is sufficiently weak or when the region of interaction is sufficiently ahead of the exit section. At present, the authors have no experimental information to assess this matter. Therefore, in applying Eqs. (17) and (18), the authors restrict themselves only to those cases where the shock/boundary-layer interaction is expected to be negligible.

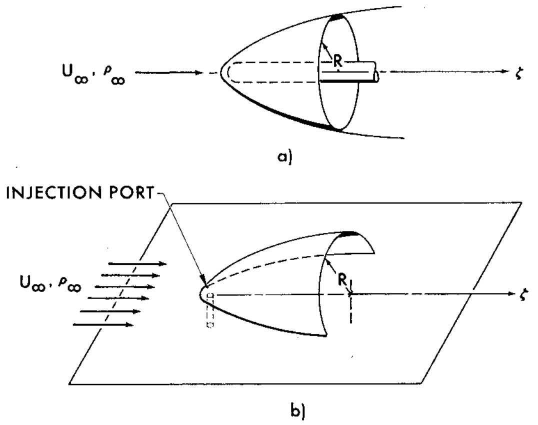

Fig. 2 Blast wave analogy.

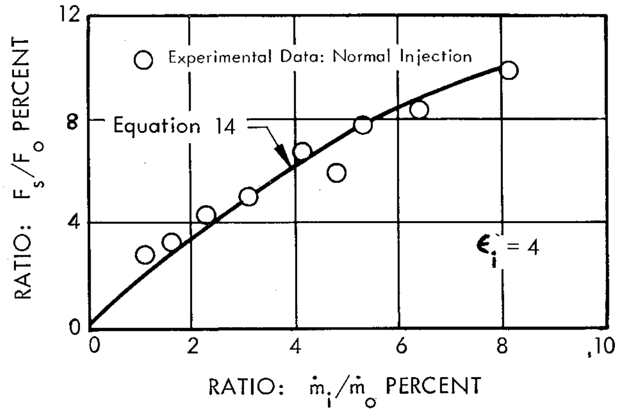

Fig. 3 Side force vs injectant mass flow rate for a 16:1 nozzle; $\epsilon_i = 4$ (experiment—Ref. 3).

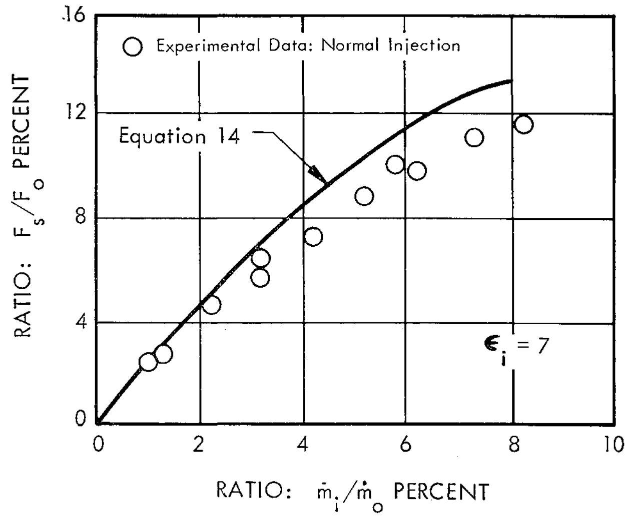

Fig. 4 Side force vs injectant mass flow rate for a 16:1 nozzle; $\epsilon_i = 7$ (experiment—Ref. 3).

Experimental Verification

Some experimental verification of the present approach is sought now. What the authors wish to do is to examine how the values of the side force F_s, as computed on the basis of Eqs. (14) and (18), compare with those measured in appropriate experiments. By appropriate experiments is meant those in which the conditions of the experiments are nearly similar to the ones stipulated in the present analysis and in which all the necessary information with regard to the given parameters of the problem is fully known. Some of the experiments of Rodriguez[3] appear to be most suitable for the purpose here. These experiments were carried out with two contoured nozzles, one with an expansion ratio of 16:1 and the other with an expansion ratio of 25:1. The primary and secondary fluids were air at room temperature. The tests

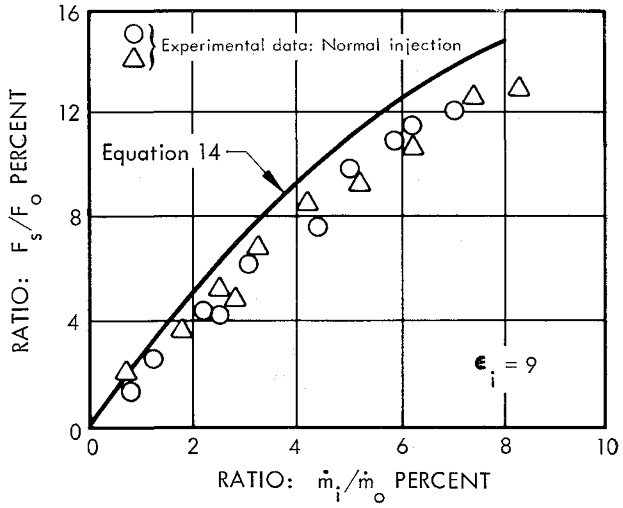

Fig. 5 Side force vs injectant mass flow rate for a 16:1 nozzle; $\epsilon_i = 9$ (experiment—Ref. 3).

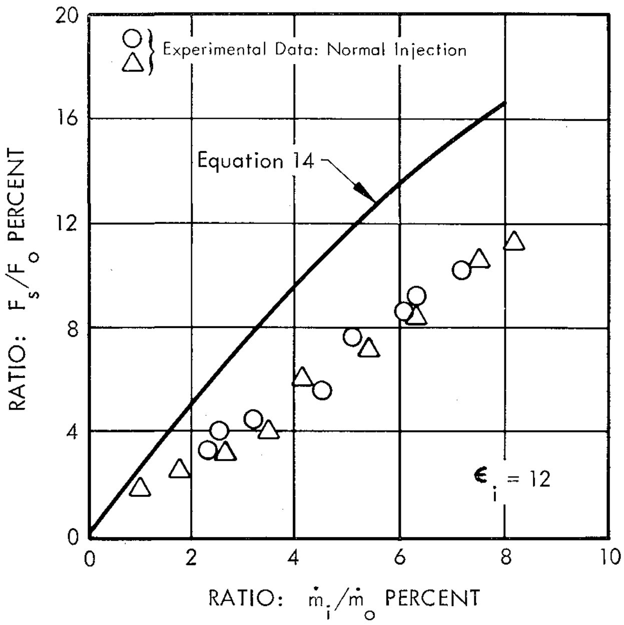

Fig. 6 Side force vs injectant mass flow rate for a 16:1 nozzle; $\epsilon_i = 12$ (experiment—Ref. 3).

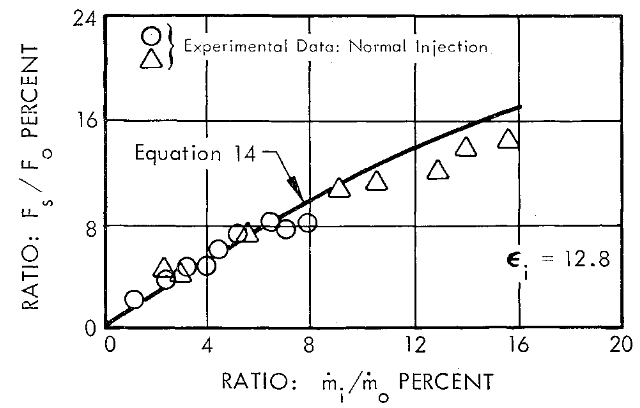

Fig. 7 Side force vs injectant mass flow rate for a 25:1 nozzle; ϵ_i = 12.8 (experiment—Ref. 3).

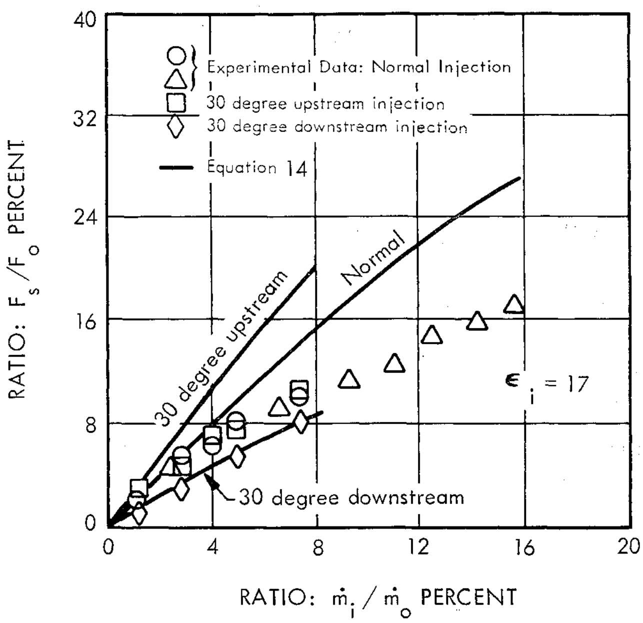

Fig. 8 Side force vs injectant mass flow rate for a 25:1 nozzle; ϵ_i = 17.1 (experiment—Ref. 3).

were done for different locations of the injection port and for different conditions of the secondary flow. Different stagnation pressures were employed for the primary and secondary flows. Secondary injection was sonic flow through a single port. The size of the injection port was varied. Most of the measurements were made for injection normal to the nozzle wall.

The computation of the side force on the basis of Eq. (14) is carried out as follows: from the conditions of the experiment all quantities, except p, in the right-hand side of Eq. (14) are calculated, A_S being calculated from Eq. (18). Since an explicit formula for p is not developed, it is computed on the basis of Eq. (7b) by using the value of the thrust augmentation δF_a measured in the experiment at the conditions under consideration.

The comparison between the computed and measured values of the side force is presented in Figs. 3–9. In these figures, F_0 is the thrust on the rocket in the absence of secondary injection, ϵ is the nozzle expansion ratio, and ϵ_i is the expansion ratio at the location of the injection port. The solid curve gives the values calculated from Eq. (14). The experimental values are shown by points. In calculating the ratio F_s/F_0 the authors have set F_0 as equal to $\dot{m}_0 U_e$, since the measured values of F_0 were not available.

Figures 3–6 all refer to the nozzle with an expansion ratio of 16. Each figure refers to a different location of the injection port. It is seen that, for ϵ_i equal to 4, 7, and 9 (Figs. 3–5), the results given by Eq. (14) agree closely with the

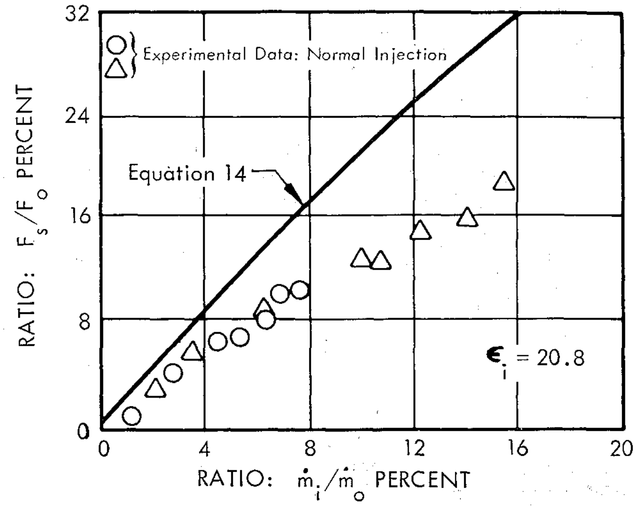

Fig. 9 Side force vs injectant mass flow rate for a 25:1 nozzle; ϵ_i = 20.8 (experiment—Ref. 3).

measured values. When the injection port is located closer to the exit section, as seen in Fig. 6, there is no agreement (as to be expected) between the calculated and measured values.

Similar results are exhibited in Figs. 7–9, which refer to the nozzle with an expansion ratio of 25. It is seen (Fig. 7) that, for ϵ_i equal to 12.8, there is close agreement between the calculated and measured values. It is interesting to note (Fig. 8) that, for ϵ_i equal to 17, there is close agreement between the calculated and measured values for 30° downstream injection although there is no agreement between them for normal and 30° upstream injection. This is perhaps due to boundary-layer shock interaction that is likely to be more predominant for the normal and upstream injection than for the downstream injection.

The disagreement exhibited in some of the figures is not disheartening, for in view of the assumptions and approximations involved in obtaining Eq. (14), one does not expect it to be valid when the injection port is near the exit section, or when the effects of shock/boundary-layer interaction become important. Under these conditions, the assumptions that the flow conditions over the area A_S are uniform in the presence of injection and that the velocity component u is approximately equal to U_e are unreasonable. Also, the computation of the area A_S by application of the blast wave analogy (apart from the assumptions implied in the analogy) becomes questionable.

It is encouraging to find that in the range of the flow conditions envisaged in the analysis, the calculated and experimental results are in close agreement. A point to be noted in this connection is that the results from Eq. (14) show that the ratio F_s/F_0 levels off at high values of the ratio $\dot{m}_i/\dot{m}_0$, apparently a result not indicated by analyses based on approaches different from the present.

Remarks

The approach given in this paper appears to offer a useful and valid method for an approximate analysis of thrust vector control by secondary fluid injection. Since the method involves a knowledge of the flow conditions only at the exit section of the nozzle, it is expected that satisfactory results may be obtained without entering into the actual details of the complex interaction flow field near the injection port and the integration of pressures over the nozzle wall.

To establish fully the method given here, further investigations suggested in the course of the analysis are needed. Of particular importance are experimental investigations. One important investigation is the determination of the shock shape at the exit section of the nozzle. Another is the measurement of the flow conditions at the exit section. The determination of the effects of shock/boundary-layer interaction in the nozzle on the shock shape and flow conditions at the exit section are also needed.

Extension of the present approach to the case of reacting fluids would again lead to a method of analyzing thrust vector control without seeking many of the details of a much more complicated flow field. (This extension is now under study.) The authors expect that, in the case of the reacting fluids, just as in the present case of nonreacting fluids, the problem would reduce to the determination of a few variables instead of that of all the variables. The solution for these may then be attempted on the basis of some over-all considerations or by suitably combining theoretical and experimental results.

By analyzing thrust vector control on the basis of the present approach, one may be able to pick out the important nondimensional parameters in the problem and know, to a satisfactory extent, the functional dependence of the side force on those parameters. For the case of inert gases, it can be shown, on the basis of Eqs. (14, 18, and 19), that the nondimensional side force is given by a relation of the form

$$\frac{F_s}{\dot{m}_0 U_e} = f\left(\frac{\dot{m}_i}{\dot{m}_0}, \frac{H_i}{U_0^2}, \frac{H_0}{U_e^2}, \gamma_0, \frac{\gamma_i}{\gamma_0}, \frac{\mu_i}{\mu_0}, \frac{p_e A_e}{\dot{m}_0 U_e}, \frac{\epsilon_i}{\epsilon}, \frac{l}{d}\right)$$

where γ denotes specific heat ratio, and μ denotes molecular weight.

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Optimal Programming Problems with Inequality Constraints I: Necessary Conditions for Extremal Solutions

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The necessary conditions are presented for an extremal solution to a programming problem with an inequality constraint on a function of the control and/or the state variables. It is shown that, in general, certain terms must be added to the Euler-Lagrange equations during intervals in which the solution curve lies on the boundary. Furthermore, for an inequality constraint function *not* explicitly involving the control variable(s), one or more functions of the state and time must satisfy equality constraints at the beginning (the entry corner) of an inequality constraint boundary interval. These constraints cause discontinuities in the influence functions (Lagrange multiplier functions) at the entry corner. The derivation of the necessary conditions which is given may also be used to allow the equations of motion to be discontinuous or even integrably infinite functions of the state as well as the time at a finite number of points. Two analytic example problems with state variable inequality constraints are presented.

1. Introduction

IN the calculus of variations, the problem of Bolza (the Mayer formulation is used) has been and continues to be of major significance. A dynamical system is considered which is represented at any time by the values of its state variables and whose development in time is determined by choices of control variable program(s). The Bolza problem asks for that control variable program(s) which will maximize (minimize) a given function of the state, while constraining other functions to specified values, at the terminal point. In this paper, the primary concerns are the modifications and additions to the necessary conditions for an extremal solution of the Bolza problem when there is an inequality constraint imposed, along the entire path, upon some function of the control and/or state variables.

Problems involving inequality constraint(s) on the control variable(s) were treated as early as 1937 by Valentine.[1] More recently, they were discussed by Cicala[2] and by Breakwell.[3] Problems involving inequality constraints on a function of the state variables with no explicit dependence on the control variables have been treated only in recent years. Gamkrelidze[4] in 1960 presented necessary conditions for extremal solutions assuming that the time derivative of the inequality constraint function was an explicit function of the control variable(s). Berkovitz[5] obtained essentially equiva-

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